Linear Time Algorithms Based on Multilevel Prefix Tree for Finding Shortest Path with Positive Weights and Minimum Spanning Tree in a Networks *

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Abstract

In this paper I present general outlook on questions relevant to the basic graph algorithms; Finding the Shortest Path with Positive Weights and Minimum Spanning Tree. I will show so far known solution set of basic graph problems and present my own. My solutions to graph problems are characterized by their linear worst-case time complexity. It should be noticed that the algorithms which compute the Shortest Path and Minimum Spanning Tree problems not only analyze the weight of arcs (which is the main and often the only criterion of solution hitherto known algorithms) but also in case of identical path weights they select this path which walks through as few vertices as possible. I have presented algorithms which use priority queue based on multilevel prefix tree – PTrie. PTrie is a clever combination of the idea of prefix tree – Trie, the structure of logarithmic time complexity for insert and remove operations, doubly linked list and queues. In C++ I will implement linear worst-case time algorithm computing the Single-Destination Shortest-Paths problem and I will explain its usage.

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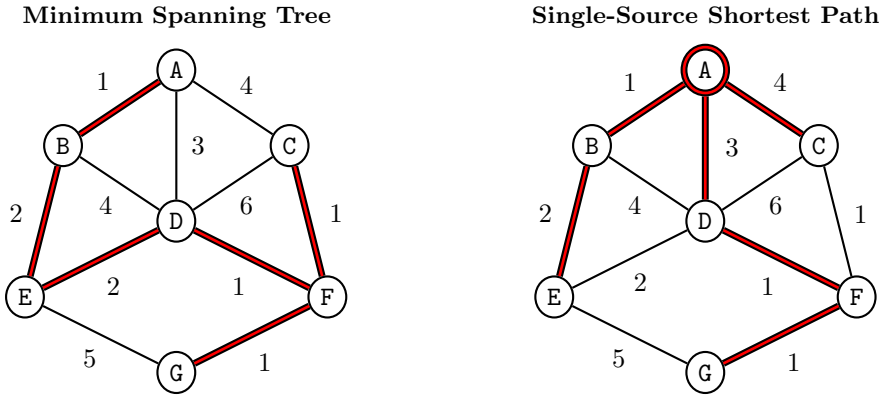
1 Introduction

Graphs are a pervasive data structure in computer science and algorithms for working with them are fundamental to the field. There are hundreds of interesting computational problems defined in terms of graph. A lot of really complex processes can be solved in a very effective and clear way by means of terms of graph. Algorithms which solve graph problems are implemented in many appliances of everyday use. They help flight control system to administer the airspace. They are crucial for economists to do market research, mathematicians to solve complicated problems. And finally, they help programmers describe object connections. Every day, many people trust graph algorithms when they, implemented in GPS system, calculate the shortest way to their destination. There are many basic graph algorithms, whose computational complexity is of greatest importance. They include algorithms on directed graphs finding Single-Source Shortest Path with positive weights (SSSP) and Minimum Spanning Tree (MST) [Figure 1]. Based on multilevel prefix tree (PTrie [28]) I compute these problems in linear worst-case time and in case of identical path weights it selects those paths which walk through as few vertices as possible.

Figure 1: Difference between MST and SSSP problems



1.1 Previous work about MST

Algorithm computing MST problem is frequently used by administrators, who think how to construct the framework of their networks to connect all servers in which they use as little optical fiber as possible. Not only computer engineers use algorithms based on MST. Architecture, electronics and many other different areas take advantage of algorithms using MST. The MST problem is one of the oldest and most basic graph problems in computer science. The first MST algorithm was discovered by Borůvka [5] in 1926 (see [26] for an English translation). In fact, MST is perhaps

the oldest open problem in computer science. Kruskal's algorithm was reported by Kruskal [25] in 1956. The algorithm commonly known as Prim's algorithm was indeed invented by Prim [31] in 1957, but it was also invented earlier by Vojtech Jarník in 1930. Effective notation of these algorithms require $O(|V|log|E|)$ time, where $|V|$ and $|E|$ denote, respectively, the number of vertices and edges in the graph. In 1975, Yao [35] first improved MST to $O(|E|loglog|V|)$, which starts with all nodes as fragments, extends each fragment, then combines, then extends each of the new enlarged fragments, then combines again, and so forth. In 1985, using a combination of the ideas from Prim's algorithm, Kruskal's algorithm and Borůvka's algorithm, together, Fredman and Tarjan [15] give on algorithm that runs in $O(|E|\beta(|E|,|V|))$ using Fibonacci heaps, where $\beta(|E|,|V|)$ is the number of log iterations on $|V|$ needed to make it less than $\frac{|E|}{|V|}$. As an alternative to Fibonacci heaps we can use insignificantly improved Relaxed heaps [12]. Relaxed heaps also have some advantages over Fibonacci heaps in parallel algorithms. Shortly after, Gabow, Galil, Spencer, and Tarjan [17] improved this algorithm to run in $O(|E|log\beta(|E|,|V|)$. In 1999, Chazelle [7] takes a significant step towards a solution and charts out a new line of attack, gives on algorithm that runs in $O(|E|\alpha(|E|,|V|))$ time, where $\alpha(|E|,|V|)$ is the function inverse of Ackermann's function [33]. Unlike previous algorithms, Chazelle's algorithm does not follow the greedy method. In 1994, Fredman and Willard [16] showed how to find a minimum spanning tree in $O(|V|+|E|)$ time using a deterministic algorithm that is not comparison based. Their algorithm assumes that the data are b-bit integers and that the computer memory consists of addressable b-bit words.

A great many of so far invented implementations attain linear time on average runs but their worst-case time complexity is higher. The invention of versatile and practice algorithm running in linear worst-case time still remains an open problem. For decades many researchers have been trying to do find linear worst-case time algorithm solving MST problem. To find this algorithm researchers start with Boruvka's algorithm and attempt to make it run in linear worst-case time by focusing on the structures used by the algorithm.

1.2 Previous work about SSSP

The Single-Source Shortest Path on directed graph with positive weight (SSSP) is one of the most basic graph problems in theoretical computer science. This problem is also one of the most natural network optimization problems and occurs widely in practice. SSSP problem consists in finding the shortest (minimum-weight) path from the source vertex to every other vertex in the graph. The Shortest Paths algorithms typically rely on the property that the shortest path between two vertices contains other shortest paths within it. Algorithms computing SSSP problem are used in considerable amount of applications. Starting from rocket software and finishing with GPS inside our cars. In many programs algorithm computing SSSP problem is a part of basic data analysis. For example, itineraries, flight schedules and other transport systems can by presented as networks, in which various shortest path problems are very important. We many aim at making the time of flight between two cities as short as

possible or at minimizing the costs. In such networks the casts may concern time, money or some other resources. In these networks particular resources don't have to be dependent. It should be noted that in reality price of the ticket may not be a simple function of the distance between two cities - it is quite common to travel cheaper by taking a roundabout route nether than a direct one. Such difficulties can be overcome by means of algorithms solving the shortest path problems. Algorithm computing SSSP problem are often used in real time systems, where is of great importance every second. Like OSPF (open shortest path first) [27] is a well known real-world implementation of SSSP algorithm used in internet routing. That's why time the efficiency of algorithms computing SSSP problem is very important. By reversing the direction of each edge in the graph, we receive Single-Destination Shortest-Paths problem (SDSP); the Shortest Path to a given destination source vertex from each vertex.

Dijkstra's algorithm was invented by Dijkstra [11] in 1959, but it contained no mention of priority queue, needs $O(|V|^2 + |E|)$ time. The running time of Dijkstra's algorithm depends on how the min-priority queue is implemented. If the graph is sufficiently sparse-in particular, $E = o(\frac{|V|^2}{lg|V|})$ - it is practical to implement the min-priority queue with a binary min-heap. Then the time of the algorithm [22] is $O(|E|log|V|)$. In fact, we can achieve a running time of $O(|V|lg|V| + |E|)$ by implementing the min-priority queue with Fibonacci heap [15]. Historically, the development of Fibonacci heaps was motivated by the observation that in Dijkstra's algorithm there are, typically, many more decrease-key calls than extract-min calls, so any method of reducing the amortized time of each decrease-key operation to $o(lg|V|)$ without increasing the amortized time of extract-min would yield on asymptotically faster implementation than with binary heaps. But Goldberg and Tarjan [20] observed in practice and helped to explain why Dijkstra's codes based on binary heaps perform better than the ones based on Fibonacci heaps. A number of faster algorithms have been developed on more powerful RAM (random access machine) model. In 1990, Ahuja, Mehlhorn, Orlin, and Tarjan [1] give on algorithm that runs in $O(|E| + |V|\sqrt{logW})$, where W is the largest weight of any edge in graph. In 2000, Thorup [34] gives on $O(|V| + |E|loglog|V|)$ time algorithm. Faster approaches for somewhat denser graphs have been proposed by Raman [32] in 1996. Raman's algorithm require $O(|E| + |V|\sqrt{log|V|loglog|V|})$ and $O(|E|+|V|(W*log|V|)^{1/3})$ time, respectively. However Asano [2] shows, the algorithms don't perform well in practical simulations. The classic label-correcting algorithm of Bellman-Ford is based on separate algorithms by Bellman [3], published in 1958, and Ford [14], published in 1956 [13], and all of its improved derivatives [8][10][30][19][4] need $\Omega(|V| * |E|)$ time in worst time. However in case of graph with irrationally heavy weight of edges algorithm's may possibly equal $O(2^{|V|})$ time cost [18]. But Bellman-Ford algorithm not only computes the single-source shortest path with positive weights, but also solves the single-source shortest path problem in the general case in which edge weights may be negative. The Bellman-Ford algorithm returns a boolean value indicating whether or not there is a negative-weight cycle that is reachable from the source. If there is such a cycle, the algorithm indicates that no solution exists. If there is no such cycle, the algorithm produces the Shortest Paths and their weights.

2 Linear worst-case time algorithms based on multilevel prefix tree computing the basic network problems

I show algorithms which use priority queue based on multilevel prefix tree – PTrie [28]. PTrie is a clever combination of the idea of prefix tree - Trie [6], the structure of logarithmic time complexity for insert and remove operations, doubly linked list [23] and queues [23].

I assume that algorithms which I present the weight of edges is constant. The weight of edges are in $\{0, \ldots, 2^t - 1\}$, where t denotes the word length (size of). For all edges of graph $G = (V, E)$ the constant value t_{max} can be matched. In other words, I assume that the size of type which remembers the weight of edges is constant and identical for all edges of graph $G = (V, E)$.

2.1 PTrie: Priority queue based on multilevel prefix tree

Priority Trie (PTrie) uses a few structures including Trie of 2^K degree [6], which is the structure core. Data recording in PTrie consists in breaking the word into parts which make the indexes of the following layers in the structure (table look-at). The last layers contain the addresses of doubly linked list's nodes. Each of the list nodes stores the queue [23], into which the elements are inserted. Moreover, each layer contains the structure of logarithmic time complexity of insert and remove operations. Which help to define the destination of data in the doubly linked list [23].

2.1.1 Terminology

Bit pattern is a set of K bits. K (length of bit pattern) defines the number of bits which are cut off the binary word. M defines number (length) of bits in a binary word.

$$\text{value of word} = \overbrace{\underbrace{101...1}_{K}00101...}^{M}$$

N is number of all values of PTrie. 2^K is variation K of element binary set $\{0, 1\}$. It determines the number of groups (number of Layers [Figure 2]), which the bit pattern may be divided into during one step (one level). The set of values decomposed into the group by the first K bits (the version of algorithm described in paper was implemented by machine of little-endian type). The path is defined starting from the most important bits of variable. The value of pattern K (index) determines the layer we move to [Figure 3]. The lowest layers determine the nodes of the list which store the queues for inserted values. L defines the level the layer is on. Probability that exactly G keys correspond to one particular pattern, where for each of P_L sequences of leading bits there is such a node that corresponds to at least two keys equals

$$\binom{N}{G} P^{-GL} (1 - P^{-L})^{N-G}$$

Figure 2: Layer

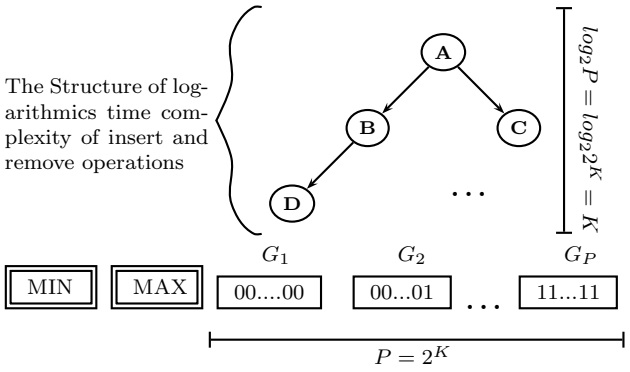
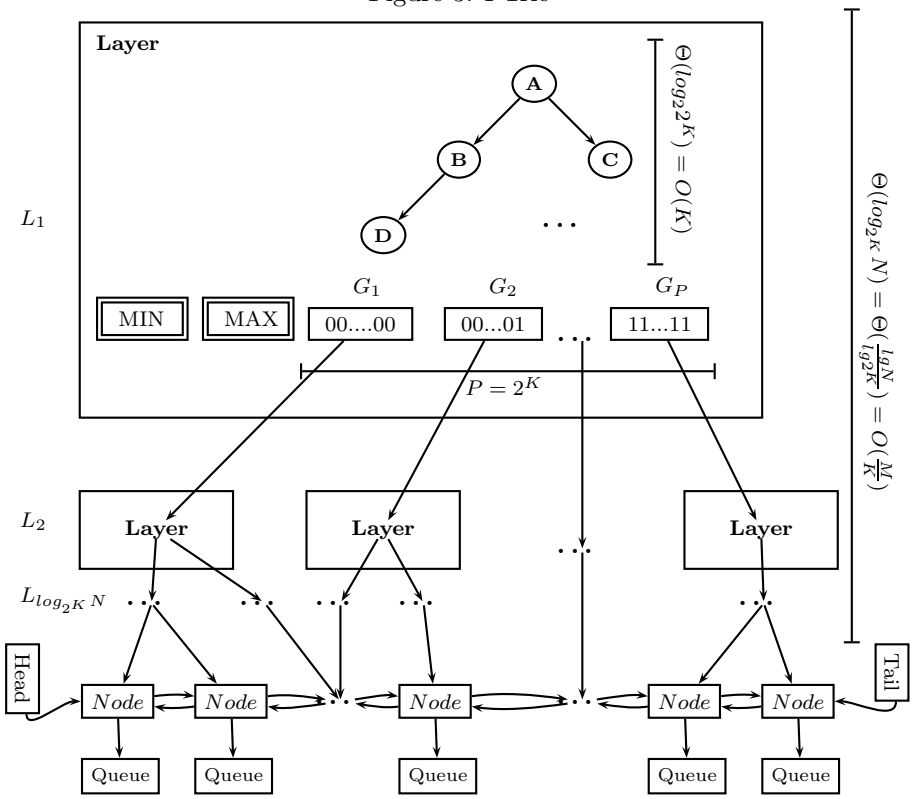


Figure 3: PTrie



For random PTrie the average number of layers on level L, for $L = 0, 1, 2, \ldots$is

$$P^L(1 - (1 - P^{-L})^N) - N(1 - P^{-L})^{N-1}$$

If A_N is average number of layers in random PTrie of degree $P = 2^K$ containing N keys. Then $A_0 = A_1 = 0$, and for $N \geq 2$ we get [24]:

$$A_N = 1 + \sum_{G_1+\ldots+G_P=N} \Big(\frac{N!}{G_1!\ldots G_P!}P^{-N}\Big)\Big(A_{G_1} + \ldots + A_{G_P}\Big) =$$

$$1 + P^{1-N} \sum_{G_1+\ldots+G_P=N} \Big(\frac{N!}{G_1!\ldots G_P!}\Big)A_{G_1} =$$

$$1 + P^{1-N} \sum_{G} \binom{N}{G}\Big(P-1\Big)^{N-G} A_G =$$

$$1 + 2^{G(1-N)} \sum_{G} \binom{N}{G}\Big(2^G-1\Big)^{N-G} A_G$$

2.1.2 Implementation

Operation	Description	Bound
create	Creates object	$O(1)$
insert(data)	Adds element to the structure.	$O(\frac{M}{K} + K)$
boolean remove(data)	Removes value from the tree. If operation failed because there was no such value in the tree it returns FALSE(0), otherwise returns TRUE(0).	$O(\frac{M}{K} + K)$
boolean search(data)	Looks for the words in the tree. If finds return TRUE(1), otherwise FALSE(0).	$O(\frac{M}{K})$
*minimum()	Returns the address of the lowest value in the tree, or empty address if the operation failed because the tree was empty.	$O(1)$
*maximum()	Returns the address of the highest value in the tree or empty address if the operation failed because the tree was empty.	$O(1)$
next	Returns the address of the next node in the tree or empty address if value transmitted in parameter was the greatest. The order of moving to successive elements is fixed - from the smallest to the largest and from "the youngest to the oldest" (stable) in case of identical words.	$O(1)$
prev	Similar to 'next' but it returns the address of preceding node in the tree.	$O(1)$

Basic operations can be joined. For example, the effect connected with the heap; delete-min() can be replaced by operations remove(minimum()).

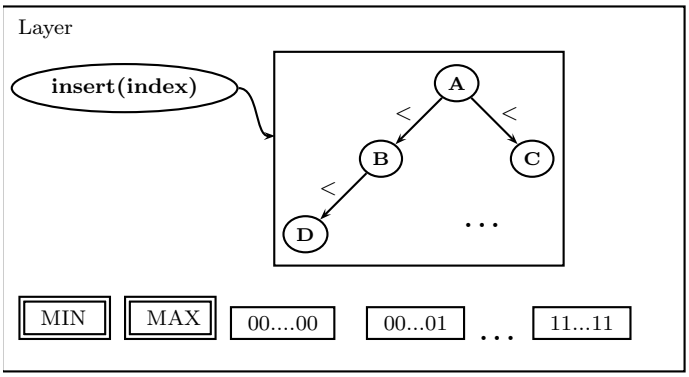
Insert
Determine the interlinked index (pointer) to another layer using the length of pattern projecting on the word.

If interlink determined by index is not empty and indicated the list node – try to insert the value into the queue of determined node.
If the elements in the queue turn out to be the same, insert value into the queue. *Otherwise*, if elements in the queue are different from the inserted value, the node is "pushed" to a lower level and the hitherto existing level (the place of node) is complemented with a new layer. Next, try again to insert the element, this time however, into the newly created layer.

Else, if the interlink determined by index is empty, insert value of index into the ordered binary tree from the current layer [Figure 4]. Father of a newly created node

in ordered binary tree from the current layer determines the place for leaves; If the newly created node in ordered binary tree is on the right side of father (added index $>$ father index), the value added to the list will be inserted after the node determined by father index and the path of the highest indexes (make use of pointer 'max' of the layers – time cost $O(1)$) of lower level layers. If newly created node is on the left side of father (added index $<$ father index), the value added to the list will be inserted before the node determined by father index and the path of the smallest indexes (make use of pointer 'min' of the layers – time cost $O(1)$) of lower level layers.
One can wonder why we use the queue and not the stack or the value counter. Value

Figure 4: Insert value of index into the ordered binary tree from the layer



counter cannot be used because complex elements can be inserted into PTrie structure, distinguishable in the tree only because of some words. Also, it is not a good idea to use a stack because the queue makes the structure stable. And this is a very useful characteristic. I used "plain" Binary Search Tree in the structure of logarithmic time complexity. For a small number of tree nodes it is a very good solution because for $K = 4$, $2^K = 16$. So in the tree there may be maximum 16 (different) elements. For such a small amount of (different) values the remaining ordered trees will probably turn out to be at most as effective as unusually simple Binary Search Trees.
Analysis: In case of random data it will take $\Theta(\frac{lgN}{lg2^K}) = \Theta(log_{2^K}N) = O(\frac{M}{K})$ goings through layers to find the place in the heap core – Trie tree. On at least one layer of PTrie structure we will use inserting into the ordered binary tree in which maximum number of nodes is 2^K. While inserting the new value I need information where exactly it will be located in the list. Such information can be obtained in two ways; I will get the information if the representation of the nearest index on the list is to the left or to the right side of the inserted word index. It may happen that in the structure there is already is exactly the same word as the inserted one. In such case value index won't be inserted into any layer of the PTrie because it will not be necessary to add a new node of the list. Value will be inserted into the queue of already existing node. To sum up, while moving through the layers of PTrie we can stop at some level because of empty index. Then, a node will be added to the list in place determined by binary search

tree and the remaining part of the path. This is why the bound of operation which inserts new value into PTrie equals $\Theta(log_{2^K} N + log_2 2^K) = \Theta(log_{2^K} N + K) = O(\frac{M}{K} + K)$.

Find
Method find like in case of plain Trie trees goes through succeeding layers following the path determined by binary representation of search value. It can be stated that it uses number key as a guide while moving down the core of PTrie – prefix tree. In case of searching tree things can happen:

- We don't reach the node of the list because the index we determine is empty on any of layers – searching failure.
- We reach the node but values from the queue are different from the searched value – searching failure.
- We reach the node and the values from the queue are exactly like the ones we seek – searching success.

Analysis: Searching in prefix tree is very fast because it finds the words using word key as indexes. In case of search failure the longest match of a searched word is found. It must be taken into consideration that during operation 'search' we use only the attributes of prefix tree. This is why the amount of search numbers looked through during the random search is $\Theta(log_{2^K} N) = O(\frac{M}{K})$.

Remove
Remove method just like find method "moves down" the PTrie structure to seek for the element to be deleted. If it doesn't reach the node of the list, or it does but the search value is different from the value of node queue, it does not delete any element of PTrie because it is not there. However if it reaches the node of the list and search value turns out to be the value from the queue – it removes the value from the queue. If it remains empty after removing the element from the queue the node will be removed from the list and will return to the "upper" layers of prefix tree to delete possible, remaining, empty layers.
Analysis: Since it is possible not only to go down the tree but also come back upwards (in case of deleting of the lower layer or the node of the list) the total length of the path move on is limited $\Theta(2log_{2^K} N)$. If delete the layer, it means there was only one way down from that layer, which implicates the fact that the ordered binary tree of a given layer contained only one node (index). The layer is removed if it remains empty after the removal of node from ordered binary tree. So the number of operation necessary for the removal of the layer containing one element equals $\Theta(1)$. In case of removal of layer L_i, if ordered binary tree of higher level layer L_{i-1}, despite removing the node which determines empty layer we came from, does not remain empty it means that there could be maximum 2^K nodes in the ordered binary tree. Operation of value delete from ordered binary tree amounts to $\Theta(log_2 2^K) = \Theta(K)$. There is no point of "climbing" up the upper layers, since the layer we came from would not be empty. At this stage the method remove ends. To sum up, worse time complexity of remove operation is $\Theta(2log_{2^K} N + K) = O(\frac{M}{K} + K)$.

Extract minimum and maximum
If the list is not empty, 'minimum' reads the value pointed by the head of the list and 'Maximum' reads the value pointed by the tail of the list.
Analysis: Time complexity of operations is $\Theta(1)$.

Iterators
The nodes of the list are linked. If we know the position of one of the nodes, we have a direct access to its neighbors. The 'next' operation reads the successor of current pointed node. The 'prev' operation reads the predecessor of currently pointed node.
Analysis: Moving to the node its neighbor requires only reading of the contents of the pointer 'next' or 'prev'. Time complexity of such operations equals $\Theta(1)$.

2.1.3 Correctness

PTrie has been designed like this, so as not to assume that keys have to be positive numbers or only integers - they can be even strings (however, in most cases the weight of arcs is represented by numbers). To insert PTrie negative and positive integers I use not one PTrie, but two! One of the structures is destined exclusively for storing positive integers and the other one for storing only negative integers. The latter structure of PTrie is responsible only for negative integers - the integers are stored in reverse order on the list (for machine of little-endian type). Therefore in case of the second structure of PTrie (responsible only for negative integers) I used standard operation of PTrie: PTrie2.maximum to extract the smallest value. Also real numbers (for example in ANSI IEEE 754-1985 standard [21]) can be used of the description of the weight of arcs on condition that two interrelated structures of PTrie will be used to put off exponent and mantissa. It is possible, because implementation of PTrie [28] described by me uses queue, which makes it stable. One of the structures of PTrie serves as storage for exponent, where each of the nodes of the list will contain additional structure of PTrie to store mantissa.

2.1.4 Conclusion

Efficiency of PTrie considerably depends on the length of pattern K. K defines optional value, which is the power of two in the range [1, min(M)]. The total size of necessary memory bound is proportional to $\Theta(\frac{log_{2^K} N(2^{K+1})}{K})$ because the number of layers required to remember N random elements in PTrie of degree 2^K equals $\frac{lgN}{lgP} * P$. Moreover, each layers has tree of maximum size 2^K nodes and table of the P-elements, so the necessary memory bound equal $\Theta(log_{2^K} N * 2P) = \Theta(\frac{M}{K} * 2^{K+1})$. For data types of constant size maximum Trie tree height equals $\frac{M}{K}$. So the pessimistic operation time complexity is $O(\frac{M}{K} + K)$. For example, for four-byte numbers it is the most effective to determine the pattern $K = 4$ bits long. Then, the pessimistic number of steps necessary for the operation on the PTrie will equal $\Theta(\frac{M}{K} + K) = \frac{32}{4} + 4 = 12$. Increasing K to $K = 8$ does not increase the efficiency of the structure operation because $\Theta(\frac{M}{K} + K) = \frac{32}{8} + 8 = 12$. What is more, in will unnecessarily increase

the memory demand. A single layer consisting of $P = 2^K$ groups for $K = 8$ will contain tables $P = 2^8 = 256$ long, not when $K = 4$, only $P = 2^4 = 16$ links. For variable size data the time complexity equals $\Theta(log_{2^K} N + K)$. Moreover, the length of pattern K must be carefully matched. For example, for strings K should not be longer than 8 bits because we could accidentally read the contents from beyond the string which normally consist of one-byte sign! It is possible to record data of variable size in the structure provided each of the analyzed words will end with identical key. There are no obstacles for strings because they normally finish with "end of line" sign. Owing to the reading of word keys and going through indexes (table look-at), primary, partial operations of PTrie method are very fast. If we carefully match K with data type, PTrie will certainly serve as a really effective Priority Queue.

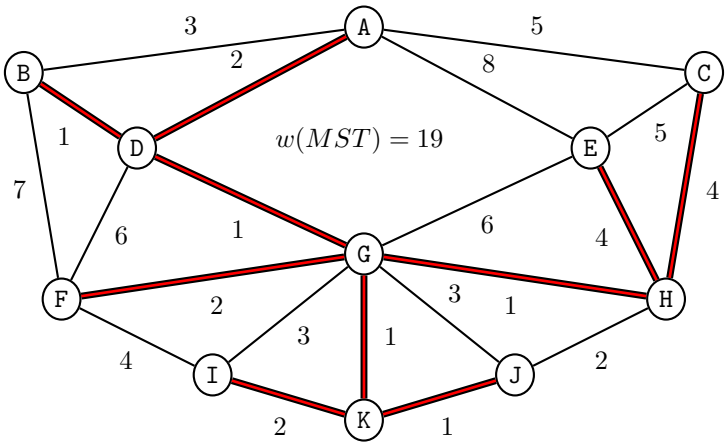
2.2 Linear time algorithm finding the Minimal Spanning Tree (MST)

Definition 2.2.1 (MST [9]) *Let $G = (V, E)$ be a connected, weighted, undirected graph. Any edges of graph $G = (V, E)$ have a weight function $w : E \to R$. Spanning tree of G is a subgraph T which contains all of the graph's vertices. The weight of a spanning tree T is the sum of the weights of its edges:*

$$w(T) = \sum_{E \in T} w(E)$$

A minimum spanning tree of $G = (V, E)$ is acyclic subset $T \subseteq E$ that connects all of the vertices and whose total weight is minimized [Figure 5].

Figure 5: The Minimum Spinning Tree of $G = (V, E)$



Let A be a subset of E that is included in some minimum spanning tree for $G = (V, E)$. Jarnik-Prim's algorithm has the property that the edges in the set A always form a single tree. The tree starts from an arbitrary source vertex s and grows until the tree spans all the vertices in V. At each step, a light edge is added to the tree A that connects A to an isolated vertex of $G_A = (V, A)$. With the proof of Jarnik-Prim's algorithm follows that by using this rule adds only edges that are safe for A; therefore, when the algorithm terminates, the edges in A form a minimum spanning tree. This strategy is greedy since the tree is augmented at each step with an edge that contributes the minimum amount possible to the tree's weight. The key to implementing Jarnik-Prim's algorithm efficiently is to make it easy to select a new edge to be added to the tree formed by the edges in A. The performance of Jarnik-Prim's algorithm depends on how we implement the min-priority queue Q. If Q is implemented as a binary min-heap, the total time for Jarnik-Prim's algorithm is $O(|V|log|V| + |E|log|V|) = O(|E|log|V|)$.

Lemma 2.2.2 *Using the priority queue based on multilevel prefix tree (PTrie) to implement the min-priority queue Q, the running time of Jarnik-Prim's algorithm improves to running worst time $O(|E| + |V|)$.*

Proof: Algorithm crosses the graph adding one edge to subset T. All the edges are inserted to PTrie – the structure working as the priority queue. In algorithm we use three operations of PTrie: insert, extract-min and decrease-key. Insert and decrease-key are characterized by $\Theta(\frac{M}{K} + K)$ time complexity, where M is the length of key required to remember the weight of edge and K is constant defined by programmers as the value of function $w(K) = (\frac{M}{K} + K)$ is minimized. Time complexity of extract-min is constant $\Theta(1)$. If we use PTrie to set a successive arcs appending to subset T, by means of Jarnik-Prim's method, we gain time complexity which amounts to $\Theta(|V|+|E|*w(k))$. Let's assume that the size of word (word length) needed to remember the weight of arcs is constant for all arcs of the graph $G = (V, E)$, then function $w(k)$ is constant. We can calculate minimum coefficient of $min\{w(k)\}$ by matching suitably K with M. Therefore time cost equals $\Theta(|V| + |E| * min\{w_{const}(k)\}) = O(|V| + |E|)$, where coefficient equals $min\{w(k) = (\frac{Mconst}{K} + K)\}$.

2.3 Minimum-weight and minimal-vertex-amount path algorithm with positive weights on directed graph in linear worst-case time (SSSP)

Definition 2.3.1 (SSSP [9]) *In a Single-Source shortest-paths with positive weights problem, we are given a weighted, directed graph $G = (V, E)$, with weight function $w : E \to R_+$ mapping edges to positive real-valued-weights. The weight of path $p =< v_0, v_1, ..., v_k >$ is the sum of the weights of its constituent edges:*

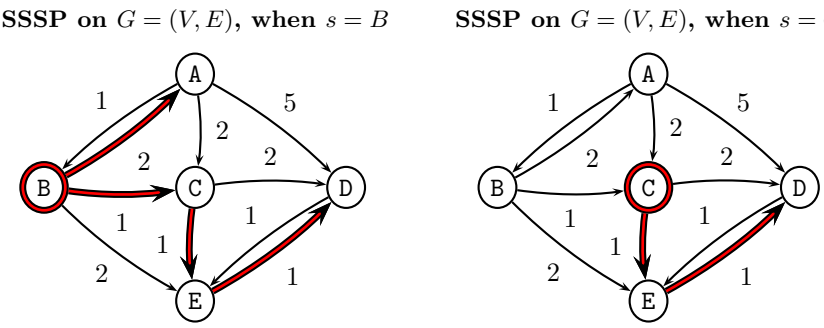
$$w(p) = \sum_{i=1}^{k} (v_{i-1}, v_i)$$

We define the shortest-path weight from u to v by

$$\delta(u,v) = \begin{cases} \texttt{min}\{w(p) : u \overset{p}{\rightsquigarrow} v\} \\ \texttt{path from u to v is not exist} \end{cases}$$

A shortest path from vertex u to vertex v is then defined as any existing path p with weight $w(p) = \delta(u,v)$ [Figure 6].

Figure 6: The Single-Source Shortest Path on directed graph $G = (V, E)$ for different source vertex s



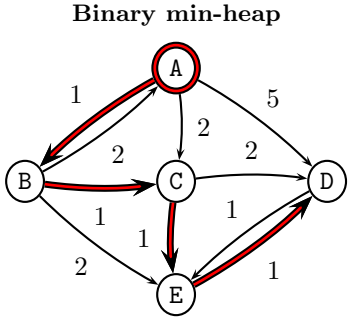
Dijkstra's algorithm maintains a set S of vertices whose final shortest-path weights from the source s have already been determined. The algorithm repeatedly selects the vertex $u \in V - S$ with the minimum shortest-path estimate, adds u to S, and relaxes all edges leaving u. The running time of Dijkstra's algorithm depends on how the min-priority queue is implemented. The performance of Dijkstra's algorithm depends on how we implement the min-priority queue Q. If Q is implemented as a binary min-heap, the total time for Dijkstra's algorithm is $O((|V| + |E|)log|V|) = O(|E|log|V|)$.

The quest for linear worst-case time Single-Source Shortest Path Algorithm on arbitrary directed graphs with positive arc weights is on ongoing hot research topic. Algorithm which I present not only finds minimum-weight path (shortest), but also makes the path walk through as few vertices as possible. I propose implementation of Dijkstra's algorithm which uses priority queue Q based on multilevel prefix tree (PTrie) [Figure 7]. PTrie is a stable structure [28]. Thanks to this algorithm it not only builds the Shortest Path of minimum-weight considering the arc weights, but also considering to the number of vertices.

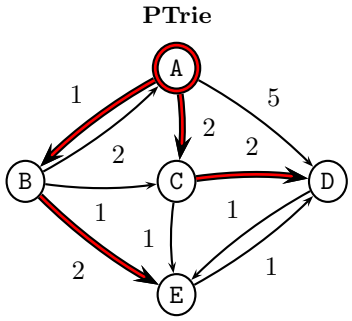
Lemma 2.3.2 *Dijkstra's algorithm where PTrie is used by priority queue request $O(|V| + |E|)$ time.*

Proof: Dijkstra's algorithm makes use of tree operations of PTrie: insert, extract-min and remove of $\Theta(|V| * |E|w(k))$ time cost. Because the length of word (size) necessary to remember the weight of arcs is constant for all arcs of graph $G =$

Figure 7: The difference of Dijkstry's algorithm between the use of basic priority queue and PTrie



path	weight	arcs
A$\overset{p}{\rightsquigarrow}$B	1	1
A$\overset{p}{\rightsquigarrow}$C	2	2
A$\overset{p}{\rightsquigarrow}$D	4	4
A$\overset{p}{\rightsquigarrow}$E	3	3



path	weight	arcs
A$\overset{p}{\rightsquigarrow}$B	1	1
A$\overset{p}{\rightsquigarrow}$C	2	1
A$\overset{p}{\rightsquigarrow}$D	4	2
A$\overset{p}{\rightsquigarrow}$E	3	2

(V, E), function $w(k)$ is constant. Function $w(k)$ is a constant coefficient which equals $min\{w(k) = (\frac{Mconst}{K} + K)\}$. Which means that time cost of particular operations executed by PTrie in case of SSSP problem equals $O(1)$. Therefore Dijkstra's algorithm where PTrie is used by priority queue needs $O(|V| + |E|)$ time.

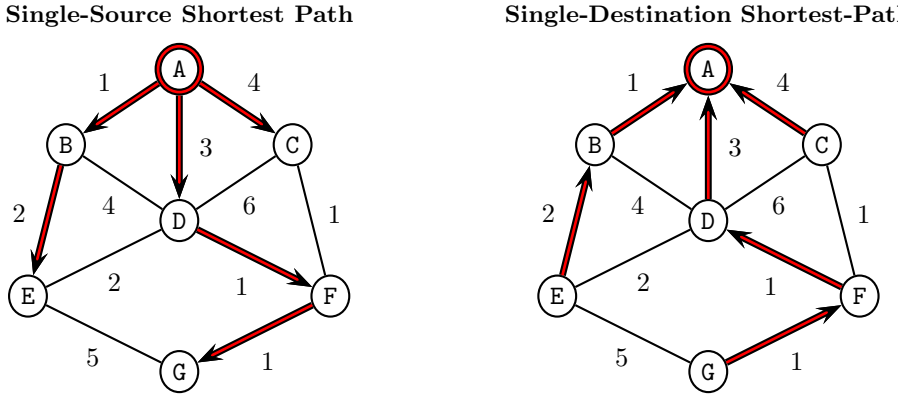
2.4 Single-Destination Shortest-Paths problem (SDSP) needs linear time

The reverse SSSP problem is commonly used in practice. Find a shortest path to a given destination source vertex s from each vertex v (SDSP). By reversing the direction of each edge in the graph $G = (V, E)$, we can reduce this problem to a single-source problem [Figure 8]. Algorithm build SDSP tree T (subset of graph) of shortest paths to source vertex from each vertex, whose leaves are all vertices $v \in V$ – without the initial source vertex $s \in V$, which is the root of T. All paths lead from each arbitrary vertex $v \in V$ to source vertex s for vertices accessible from a given destination source vertex s .

Definition 2.4.1 (SDSP) *We are given a weighted, directed graph $G = (V, E)$, represented by adjacent list, with weight function $w : E \to R_+$ mapping edges to positive real-valued-weights. The weight of path $p =< v_0, v_1, \ldots, v_k >$, is the sum of the weights of its constituent edges. For graph $G = (V, E)$ exist destination source vertex $s \in V$; for all vertices $v \in V$ it is necessary to find the shortest path with v to s.*

Similarly to other algorithms I use the property that Shortest Paths algorithms typically rely on the property that the shortest path between two vertices contains other

Figure 8: Difference between SSSP and SDSP problems



shortest paths within it. But I assume double criterion to build the shortest path. I build the shortest path relative to the weight of arcs and then relative to the amount of vertices which contain the shortest (minimum-weight and minimum-vertices) path. That is possible thanks to the stability of PTrie implementation [28]. That solution is suitable, because it may happen that there exist many shortest paths related to the weight of arcs. In these circumstances, minimal-weight path with minimum amount of arcs becomes the Shortest Path.

2.4.1 Structure of vertex and arc

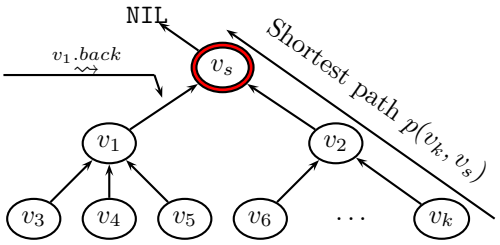
Structure of vertex contain type '*data*' which storage label of vertex. Because graph G is represented by adjacent list, each vertex has the list of pointers to the neighbors. The order on the list is random. The structure of vertex has a helpful variable '*back*' (used by any graph algorithms), which indicates one of arcs locate on the adjacent list of the structure. Each vertex $v \in V$ has a link '*back*' to the neighbor (vertex from the adjacent list), in that moment considered the successor on the shortest path, or '*back*' equals NIL. For this reason we can for example, differentiate the vertices added to SDSP tree T [Figure 9] from those ones which hasn't been analyzed yet.

```
Vertex{
data;
Neighbors *list;
Neighbors *back;
};
```

The structure of arc serves to insert information about arc to PTrie. Therefore PTrie will be able to store not only integer but detail information about arbitrary arc too. The Arc structure stores information about the weight of arc and path, and two pointers; to vertex which is the tail of the arc and to vertex which is the head of the arc. The '*pathWeight*' contains the sum of optimal arcs, which follow from the source vertex to currently

```
Arc{
weight;
pathWeight;
Vertex *tail;
Vertex *head;
};
```

Figure 9: SDSP tree



analyzed vertex. PTrie uses this variable to determine the order.

Such graph implementation allows considerable adaptability. It's enough to know the address of vertex (pointer) to get to know the shortest path to a given destination of the source vertex. And by the way meet all vertices which are located on this path.

2.4.2 Algorithm

The algorithm starts with the source vertex $s \in V$ and inserts the adjacent list of 's' to PTrie. Then with the help of '$minimum$' and '$remove$' operations '$extracts - remove - min$' of arcs from PTrie. We move on to the arc leadings to vertex. If vertex has not been attached to SDSP tree yet (the value of back is equal NIL) the algorithm will attach the vertex to SDSP tree. By setting the pointer '$back$' on the tail (vertex) of the arc which leads to the current vertex. Next, all arcs of the analyzed vertex increased by the weight of the path, which leads to the current vertex, are inserted to PTrie. Again we choose the smallest arc from PTrie... The algorithm ends its work when PTrie is empty. It means that all arcs accessible from the source vertex were browsed. Visited vertices have set arc '$back$' is such a way that the path which the arcs '$back$' built is not only the minimum-weight path (the amount of arc weights is the smallest) but also the path walks through as few arcs as possible.

Pseudo-code of algorithm compute SDSP problem

```
(1)  SDSP(G, s) begin
(2)     PTrie.insert(s.⟶neighbors)
(3)     while(PTrie is not empty) begin
(4)        arc = PTrie.minimum()
(5)        PTrie.remove(arc)
(6)        if(arc.head.back is empty and isn't s) begin
(7)           arc.head.back = reverse(arc)
(8)           PTrie.insert(arc.head.⟶neighbors + arc.pathWeight)
(9)        end
(10)    end

(11) end
```

1. The algorithm begins to build the SDSP tree from the arbitrary source vertex 's'. SDSP tree consists of all vertices accessible from any source vertex.

2. Insert the adjacent list to PTrie.

3. The algorithm will check the paths stored in PTrie as long as they exist.

4. I take and remember the path of the smallest weight from PTrie and the last arc of this path. The variable '$weightPath$' defines the weight of the whole path. The variable '$weight$' defines the weight of the last arc, where the last arc is represented by variables '$tail$' and '$head$'. The path leads from the source vertex to the vertex indicate by '$head$'

5. Remove the path of the smallest weight from PTrie.

6. If the vertex which the arc leads to has not been added to SDSP tree yet and it is not the source vertex...

7. Ascribe the reverse of analyzed arc to the supportive arc '$back$'.
   `arc:` $A \rightarrow B$
   `reverse(arc):` $B \rightarrow A$

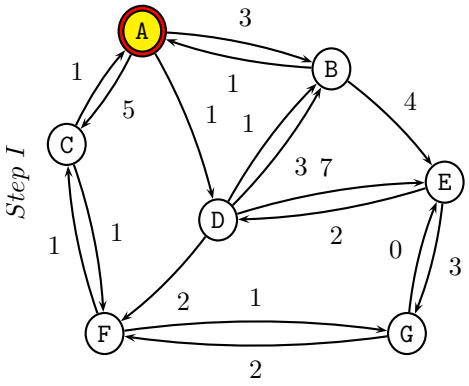
8. Insert the arc of analyzed vertex to PTrie adding the weight of the path which brought us to the analyzed vertex.

9. If the vertex has already been added to SDSP tree or its is a source vertex, it is not analyzed any more.

10. The algorithm finished checking all arcs/vertices which were accessible from the source vertex.

11. When the algorithm finishes its work an vertices accessible from the source vertex by the supportive arcs '$back$' build SDSP tree, whose root and vertex constitute the source vertex, to which lead all the paths based on the arcs '$back$'.

Figure 10: Algorithm compute SDSP of a work, step I

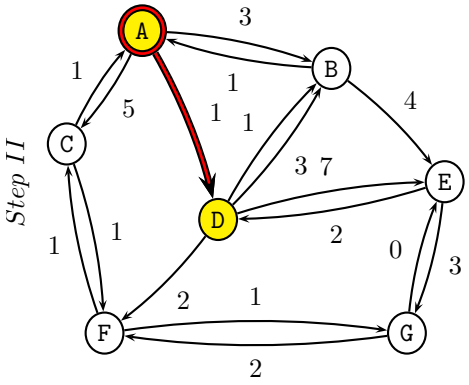


Inside the PTrie:

weight	path
1	$A \overset{p}{\rightsquigarrow} D$
3	$A \overset{p}{\rightsquigarrow} B$
5	$A \overset{p}{\rightsquigarrow} C$

extract-min: $A \overset{p}{\rightsquigarrow} D$
path weight: $w(p) = 1$
path reject: `FALSE`

Figure 11: Algorithm compute SDSP of a work, step II



Inside the PTrie:

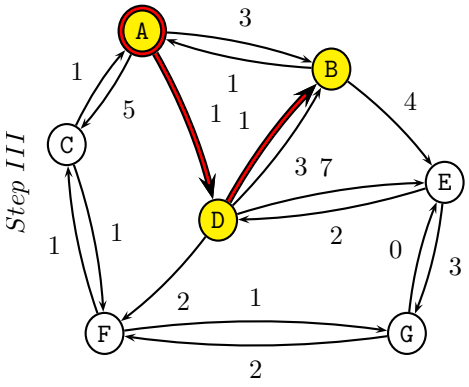
weight	path
2	$D \overset{p}{\rightsquigarrow} B$
3	$A \overset{p}{\rightsquigarrow} B$
3	$D \overset{p}{\rightsquigarrow} F$
4	$D \overset{p}{\rightsquigarrow} B$
5	$A \overset{p}{\rightsquigarrow} C$
8	$D \overset{p}{\rightsquigarrow} E$

extract-min: $D \overset{p}{\rightsquigarrow} B$
path weight: $w(p) = 2$
path reject: `FALSE`

2.4.3 Analysis of the algorithm work

I will analyze the algorithm work step by step; How and in what order arcs are inserted to PTrie? What is the sequence of vertex attachment to the tree containing the solution of SDSP problem? Step by step description of the algorithm computing SDSP problem at work [Figures 10,11,12,13,14,15,16,17,18,19,20].

Figure 12: Algorithm compute SDSP of a work, step III

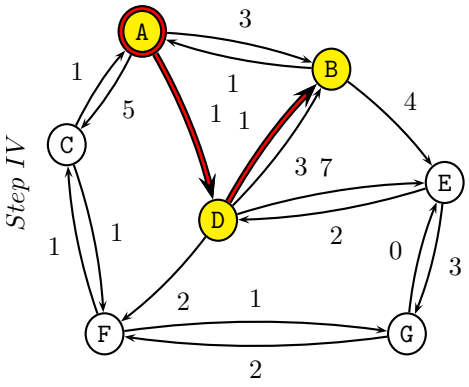


Inside the PTrie:

weight	path
3	$A \overset{p}{\rightsquigarrow} B$
3	$D \overset{p}{\rightsquigarrow} F$
3	$B \overset{p}{\rightsquigarrow} A$
4	$D \overset{p}{\rightsquigarrow} B$
5	$A \overset{p}{\rightsquigarrow} C$
6	$B \overset{p}{\rightsquigarrow} E$
8	$D \overset{p}{\rightsquigarrow} E$

extract-min: $A \overset{p}{\rightsquigarrow} B$
path weight: $w(p) = 3$
path reject: `TRUE`

Figure 13: Algorithm compute SDSP of a work, step IV

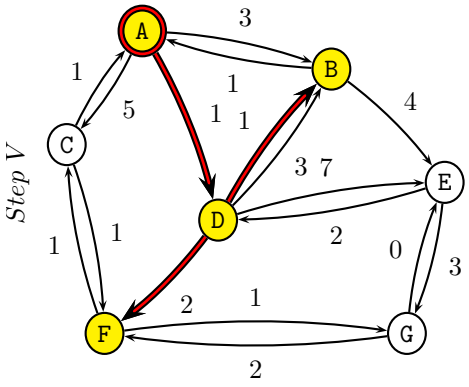


Inside the PTrie:

weight	path
3	$D \overset{p}{\rightsquigarrow} F$
3	$B \overset{p}{\rightsquigarrow} A$
4	$D \overset{p}{\rightsquigarrow} B$
5	$A \overset{p}{\rightsquigarrow} C$
6	$B \overset{p}{\rightsquigarrow} E$
8	$D \overset{p}{\rightsquigarrow} E$

extract-min: $D \overset{p}{\rightsquigarrow} F$
path weight: $w(p) = 3$
path reject: `FALSE`

Figure 14: Algorithm compute SDSP of a work, step V

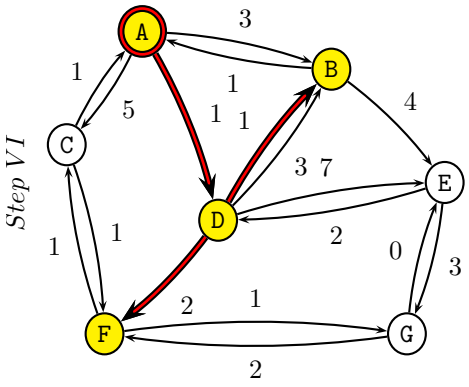


Inside the PTrie:

weight	path
3	$B \overset{p}{\rightsquigarrow} A$
4	$D \overset{p}{\rightsquigarrow} B$
4	$F \overset{p}{\rightsquigarrow} C$
4	$F \overset{p}{\rightsquigarrow} G$
5	$A \overset{p}{\rightsquigarrow} C$
6	$B \overset{p}{\rightsquigarrow} E$
8	$D \overset{p}{\rightsquigarrow} E$

extract-min: $B \overset{p}{\rightsquigarrow} A$
path weight: $w(p) = 3$
path reject: `TRUE`

Figure 15: Algorithm compute SDSP of a work, step VI

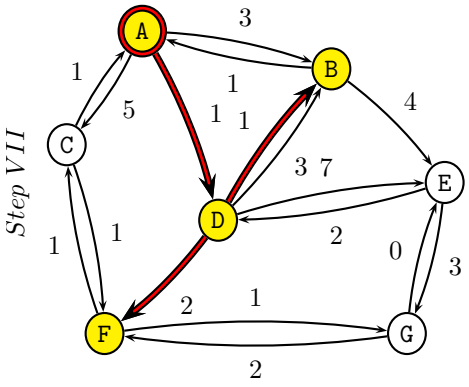


Inside the PTrie:

weight	path
4	$D \overset{p}{\rightsquigarrow} B$
4	$F \overset{p}{\rightsquigarrow} C$
4	$F \overset{p}{\rightsquigarrow} G$
5	$A \overset{p}{\rightsquigarrow} C$
6	$B \overset{p}{\rightsquigarrow} E$
8	$D \overset{p}{\rightsquigarrow} E$

extract-min: $D \overset{p}{\rightsquigarrow} B$
path weight: $w(p) = 4$
path reject: `TRUE`

Figure 16: Algorithm compute SDSP of a work, step VII

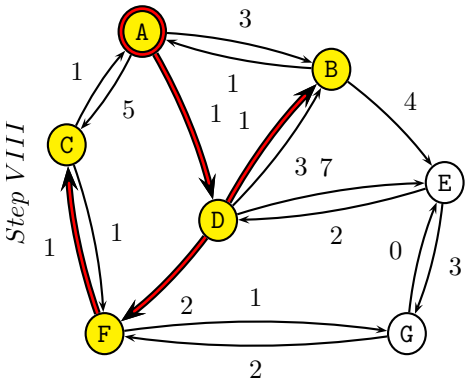


Inside the PTrie:

weight	path
4	$F \overset{p}{\rightsquigarrow} C$
4	$F \overset{p}{\rightsquigarrow} G$
5	$A \overset{p}{\rightsquigarrow} C$
6	$B \overset{p}{\rightsquigarrow} E$
8	$D \overset{p}{\rightsquigarrow} E$

extract-min: $F \overset{p}{\rightsquigarrow} C$
path weight: $w(p) = 4$
path reject: `FALSE`

Figure 17: Algorithm compute SDSP of a work, step VIII

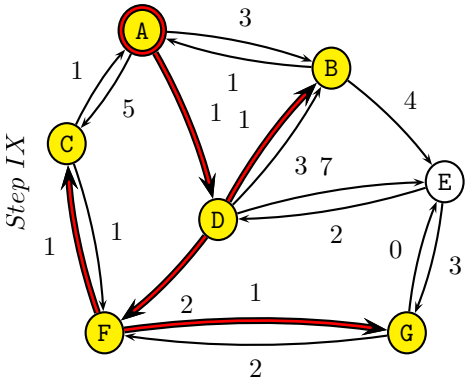


Inside the PTrie:

weight	path
4	$F \overset{p}{\rightsquigarrow} G$
5	$A \overset{p}{\rightsquigarrow} C$
5	$C \overset{p}{\rightsquigarrow} A$
5	$C \overset{p}{\rightsquigarrow} F$
6	$B \overset{p}{\rightsquigarrow} E$
8	$D \overset{p}{\rightsquigarrow} E$

extract-min: $F \overset{p}{\rightsquigarrow} G$
path weight: $w(p) = 4$
path reject: `FALSE`

Figure 18: Algorithm compute SDSP of a work, step IX

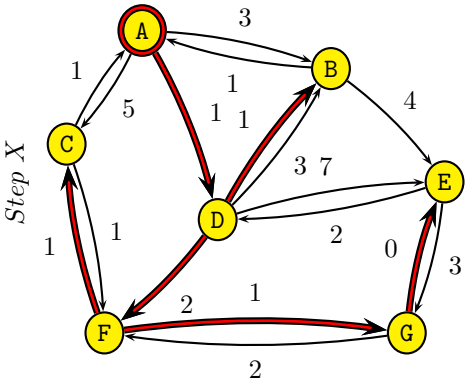


Inside the PTrie:

weight	path
4	$G \overset{p}{\rightsquigarrow} E$
5	$A \overset{p}{\rightsquigarrow} C$
5	$C \overset{p}{\rightsquigarrow} A$
5	$C \overset{p}{\rightsquigarrow} F$
6	$B \overset{p}{\rightsquigarrow} E$
8	$D \overset{p}{\rightsquigarrow} E$

extract-min: $G \overset{p}{\rightsquigarrow} E$
path weight: $w(p) = 4$
path reject: `FALSE`

Figure 19: Algorithm compute SDSP of a work, step X

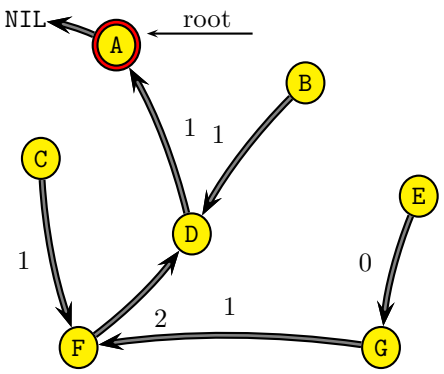


Inside the PTrie:

weight	path
5	$A \overset{p}{\rightsquigarrow} C$
5	$C \overset{p}{\rightsquigarrow} A$
5	$C \overset{p}{\rightsquigarrow} F$
6	$B \overset{p}{\rightsquigarrow} E$
6	$E \overset{p}{\rightsquigarrow} D$
7	$E \overset{p}{\rightsquigarrow} G$
8	$D \overset{p}{\rightsquigarrow} E$

extract-min: remaining
path weight:
path reject: `TRUE`

Figure 20: Algorithm compute SDSP of a work: SDSP tree



Legend: gray arcs denote pointer 'back' on vertices
yellow vertex with red aureola is a root
yellow vertices without red aureola are leaves

2.4.4 Analysis of correctness

The algorithm shown here starts to analyze the graph and create the shortest paths to the source vertex. If the graph G is not strongly connected [2] the algorithm which solves SDSP problem and starts its work from the source vertex will calculate SDSP tree of connected components [3] containing the source vertex; $G_{s \in V} = (s, V, E)$.

The algorithm does not use weight relaxation. Arcs added to the SDSP tree are not modified any more. Only the order of taking the paths out of the PTrie determines the choice of arcs or paths, starting from the minimum-weight path. It's worth remembering, however, that the weight of each vertex which has not been added to SDSP tree should be increased by the weight of the path which brought us there before inserting it to PTrie. So the vertices to which lead the minimum-weight path are visited always but only once. The SDSP tree is represented by the '*back*' connected with the vertices. That's means that for arbitrary graph $G = (V, E)$ with the directed source vertex the algorithm define the tree, which is the subgraph of the predecessor of the graph G as graph $T_{back} = (V, E_{back})$. Therefore the algorithm is correct because the shortest paths are composed of shortest paths. The proof of this is based on the notion that if there was a shorter path than any sub-path, then the shorter path should replace that sub-path to make the whole path shorter. That's why the subgraph of predecessors T_{back} is the Shortest Path Tree.

[2] A directed graph is strongly connected if every two vertices are reachable from each other.

[3] The connected components of a graph are the equivalence classes of vertices under the "is reachable from" relation.

2.4.5 Analysis of the algorithm bound

Body loop which inserts arcs to PTrie is $\Theta(|E|)$ time cost. The operation of PTrie for constant length (size of) type weight of arc are $\Theta(\frac{Mconst}{K} + K) = O(1)$. To look through each of vertex graph the algorithm require $\Theta(|V|)$ time. Therefore worst-case time complexity equals $\Theta(|V| + |E|(\frac{Mconst}{K} + K)) = O(|V| + |E|)$ time.

2.4.6 A simple example of the use of algorithm computing SDSP problem in C++

The algorithm builds SDSP tree for graph presented in [Figure 10].

```
#include<iostream>

// The Report contains the PTrie source code:
// \protect\vrule width0pt\protect\href{http://techreports.library.cornell.edu:8081/}{http://techreports.library.co
// Dienst/UI/1.0/Display/cul.cis/TR2006-2023
//
// Or, the SourceForge.net project:
// \protect\vrule width0pt\protect\href{http://ptrie.sourceforge.net}{http://ptrie.sourceforge.net}
#include "PTrie.hpp" /*
                     ** \protect\vrule width0pt\protect\href{http://ptrie.sourceforge.net/source/PTrie.hpp}{http://
                     */


// <begin> namespace dplaneta
namespace dplaneta{

struct Vertex;
struct Neighbors;
struct Arc;

/*****************************************************************************
** The definition of the structure which represents vertex.
*/
struct Vertex{
        char data; // Label
        Neighbors *list; //Adjacent List

        Vertex *back;
        unsigned backWeight;

        // The metod is responsible for connecting vertices with each other.
        Vertex& attach(Vertex *attach, unsigned weight);

        Vertex(char name);
        ~Vertex(void);
        };
```

```
/*****************************************************************************
** The definition of the structure responsible for the organization
** of adjacent list .
*/
struct Neighbors{
        Neighbors *next; // Adjacent List [the singly-linked list ]

        // Arc representation
        Vertex *link;
        unsigned weight;

        Neighbors(Vertex *l, unsigned w, Neighbors *n);
        ~Neighbors(void);
        };




Vertex::Vertex(char name): list(0), back(0), data(name){}
Vertex::~Vertex(void){
         delete list;
         back = 0;
         list  = 0;
         }


// The method 'attach' implemented in vertex structure is responsible for the
// connection of the vertices . For example, to connect vertex 'A' with 'B'
// [Vertex *a = new Vertex('A'), *b = new Vertex('B')]
// We add the edge which connects vertex A to its adjacent list .
// [a->add(b, lenght);]
// Because of this we get the arc (directed edge) joining 'A' and 'B'.
// To simulate the edge between 'A' and 'B', also vertex 'B' should contain the
// arc of the same length which joins it with 'A'. [b->add(a, lenght);]
// (A) <===> (B).
Vertex& Vertex::attach(Vertex *attach, unsigned weight){
         list  = new Neighbors(attach, weight, list);
         return *this;
}




Neighbors::Neighbors(Vertex *l, unsigned w, Neighbors *n):
            link(l), weight(w), next(n){}

Neighbors::~Neighbors(void){
```

```
                    delete next;
                    next = 0;
                    }
```

```
/*****************************************************************************
** The definition of the structure which represents arc is necessary
** inside the PTrie. The structure must have the implemented overloaded
** operators used by PTrie.
*/
struct Arc{
        unsigned weight, pathWeight;
        Vertex *tail , *head;
        unsigned operator>>(const unsigned i) const
              { return (pathWeight>>i); }

        bool operator!=(const Arc& obj) const
              { return this->pathWeight!=obj.pathWeight; }

        bool operator==(const Arc& obj) const
              { return this->pathWeight==obj.pathWeight; }
        };
```

```
/*****************************************************************************
** Supportive function used by PTrie to return size of Type<Arc.weight>
*/
inline size_t sizeFunc(const class Arc& obj){
        return sizeof(obj.weight);
}
```

```
// If You want to see how the arcs are analyzed by the algorithm...
// #define TEST

/*****************************************************************************
** The algorithm builds the SDSP tree by proper arrangement of support
** varieties 'back' which are located in every vertex.
*/
void SDSP(Vertex* sourceVertex){
      register Arc temp;
      register const Arc *p;
      register const Neighbors *t;
```

```
    PTrie<Arc> Q(sizeFunc);
    PTrie<Arc>::iterator iter=Q;

#ifdef TEST
std::cout<<"Insert the adjacent list of the source vertex to PTrie:\n";
#endif
    for(t = sourceVertex->list; t!=NULL; t=t->next){
        temp.weight = temp.pathWeight = t->weight;
        temp.tail = sourceVertex;
        temp.head = t->link;
        Q.insert(temp);
#ifdef TEST
std::cout<<"["<<sourceVertex->data<<"]--("<<t->weight<<")->["<<t->link->data<<"]\n";
#endif
    }

#ifdef TEST
std::cout<<std::endl;
#endif

    // All arcs accessible from the source vertex are analyzed.
    while(p = Q.minimum()){

#ifdef TEST
std::cout<<"\tCame from the ["<<p->tail->data<<"]; State of the PTrie: \n";
iter.begin();
while(iter){
  std::cout<<"\t["<<(*iter).tail->data<<"]--("<<(*iter).pathWeight<<")->["
           <<(*iter).head->data<<"]\n";
  iter++;
}
std::cout<<std::endl;
#endif

        if(p->head->back==NULL && p->head!=sourceVertex){

#ifdef TEST
std::cout<<"\nThe vertex is chosen: ["<<p->head->data
         <<"] and insert the adjacent list to PTrie:\n";
#endif

            ((Arc*)p)->head->back = p->tail;
            p->head->backWeight = p->weight;

            // Insert the adjacent list of the current vertex to PTrie
            for(t = p->head->list; t!=NULL; t=t->next){
                temp.weight = t->weight;
                temp.pathWeight = t->weight + p->pathWeight;
                temp.tail = p->head;
                temp.head = t->link;
```

```
                Q.insert(temp);

#ifdef TEST
std::cout<<"["<<temp.tail->data<<"]--("<<temp.weight<<")->["
          <<temp.head->data<<"]"\
" weight of path = "<<temp.pathWeight<<std::endl;
#endif
            }

#ifdef TEST
std::cout<<std::endl;
#endif
        }
        Q.remove(*p);

    }


}




// The function walks on the path from the choosen vertex to the source vertex.
void Walk(const Vertex *v){
    while(v){
            std::cout<<"["<<v->data<<"]";
            if(v->back) std::cout<<"--("<<v->backWeight<<")->";
            v = v->back;
    }
    std::cout<<std::endl;

}




// The function shows the adjacent list of the choosen vertex.
void ShowAdjecentLists(const Vertex *v){
    for(const Neighbors *t = v->list; t!=NULL; t=t->next)
        std::cout<<"\t["<<v->data<<"]--("<<t->weight<<")->["
                 <<t->link->data<<"]\n";
    std::cout<<std::endl;
}


}// <end> namespace dplaneta
```

```
/*****************************************************************************
** This example source code demonstrates how you can use a shown algorithm
** computing SDSP problem.
*/
int main(int argc, char *argv[]){
// We create the graph vertices
dplaneta::Vertex a('A'), b('B'), c('C'), d('D'), e('E'), f('F'), g('G');
dplaneta::Vertex *sourceVertex;

// Connecting the graph vertices.
a.attach(&b, 3).attach(&c, 5).attach(&d, 1);
b.attach(&a, 1).attach(&e, 4);
c.attach(&f, 1).attach(&a, 1);
d.attach(&b, 1).attach(&b, 3).attach(&e, 7).attach(&f, 2);
e.attach(&d, 2).attach(&g, 3);
f.attach(&g, 1).attach(&c, 1);
g.attach(&e, 0);

std::cout<<"Show adjecent lists:\n----------------------\n";
std::cout<<"A:\n"; ShowAdjecentLists(&a);
std::cout<<"B:\n"; ShowAdjecentLists(&b);
std::cout<<"C:\n"; ShowAdjecentLists(&c);
std::cout<<"D:\n"; ShowAdjecentLists(&d);
std::cout<<"E:\n"; ShowAdjecentLists(&e);
std::cout<<"F:\n"; ShowAdjecentLists(&f);
std::cout<<"G:\n"; ShowAdjecentLists(&g);
std::cout<<"----------------------\n\n";

// We choose on arbitral vertex from which the algorithm will build SDSP tree.
sourceVertex = &a;


// We create the SDSP tree, by suitably setting supportive
// variable 'back' which are located in each vertex.
dplaneta::SDSP(sourceVertex);

// We check the paths.
std::cout<<"SDSP Path(es?):\n";
dplaneta::Walk(&a);
dplaneta::Walk(&b);
dplaneta::Walk(&c);
dplaneta::Walk(&d);
dplaneta::Walk(&e);
dplaneta::Walk(&f);
dplaneta::Walk(&g);
```

```
return 0;
}
```

3 Conclusions

I have shown linear worst-worst case algorithms based on PTrie which compute the basic Network problems. Despite the fact that PTrie is based on digital data, it can be used to store positive integer, integer but also real numbers. Because all quantities in computer are represented by binary words. That's why the weight of arc can be defined not only by positive integer, but also by real number, or even by string. Thanks PTrie, which is stable, during computing the MST, SSSP and SDSP problems, we not only focus on the Shortest Path in relation to weight of arcs, but also to the amount of vertices and arcs on the path too. Time complexity of mentioned algorithms equals $O(|V| + |E|)$. Memory bound of algorithms equals memory bound of PTrie. PTrie memory bound equals $\Theta(\frac{log_{2^K} N(2^{K+1})}{K})$. Presented algorithms not only get the fastest asymptotic running time, but they are also very practicable and can be easily implemented.

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